Title: Inter-universal Teichmüller Theory as an Anabelian Gateway to Diophantine Geometry and Analytic Number Theory (MFO-RIMS23 Version)

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Abstract:

One question that is frequently asked concerning *inter-universal Teichmüller theory* (IUT) is the following:

Why/how does IUT allow one to apply *anabelian geometry* to prove *diophantine* results?

In this talk, we address this question from various points of view. First, we discuss the fundamental framework underlying the relationship established by IUT between anabelian geometry, on the one hand, and diophantine geometry/analytic number theory, on the other. This discussion centers around the N-th power map on a subring of a field and the difference between regarding a group as a *Galois group*, on the one hand, and as an *abstract group* that is not equipped with an embedding into the automorphism group of a field, on the other. Here, we emphasize that this discussion is *entirely elementary* and only assumes a knowledge of *groups/monoids, rings, fields*, and the elementary geometry surrounding the *projective line/Riemann sphere*. We also briefly discuss certain (again entirely elementary!) *set-theoretic/foundational subtleties* surrounding the notion of a *"gluing"*. Such subtleties include the importance of working with "types/packages of data" called *"species"* (as opposed to underlying sets!), as well as the importance of obtaining *"closed loops"* of such types/packages of data in order to establish set-theoretic conclusions. Classical instances of such subtleties include the *conjugacy indeterminacies* inherent in the construction of the *algebraic closure* of a field and the closely related use of *norms* in Galois theory, as well as the classical notions of *analytic continuation/Riemann surfaces* (which is reminiscent of the classical dispute between *Riemann* and *Weierstrass*!) and *geodesic completeness/closed geodesics*. We then proceed to survey recent developments (work in progress) in IUT, many of which are closely related to the *Section Conjecture* in anabelian geometry for arbitrary hyperbolic curves over number fields. We also briefly mention recent progress on the Section Conjecture for hyperbolic curves over *p*-adic local fields, which is of interest in that it is closely related to the use of *Raynaud-Tamagawa "new-ordinariness"* in recent results on *"RNS"* (i.e., "resolution of nonsingularities"), in a fashion that may be regarded as a sort of *p*-adic local analogue of IUT. In the case of the Section Conjecture for hyperbolic curves over number fields, recent progress is closely related to 3 new enhanced versions of IUT that are currently under development. One of these new enhanced versions, namely, the *Galois-orbit version* of IUT, has new applications not only to the *Section Conjecture* for hyperbolic curves over number fields, but also to the *nonexistence of Siegel zeroes of certain Dirichlet L-functions*. The application to the Section Conjecture is interesting in that it exhibits and reconfirms the *essentially anabelian content of IUT*, i.e., as a *theory based on anabelian geometry that is applied to prove new results in anabelian geometry*. On the other hand, these recent applications, taken together with the original application of IUT to the ABC/Szpiro/Vojta Conjectures, are also noteworthy in that they may be regarded as a striking example of Poincaré's famous quote to the

effect that

“mathematics is the art of giving the same name to different things”.

That is to say, the *common name “IUT”* that may be regarded as describing, in essence, a *single mathematical phenomenon* that manifests itself, depending on relatively inessential (!) differences of context, as various (at first glance, unrelated!) *diverse phenomena* in *anabelian geometry*, *diophantine geometry*, and *analytic number theory*. The relationship with Poincaré's famous quote is also fascinating in that it was apparently motivated by various mathematical observations on the part of Poincaré concerning the similarities between *transformation group symmetries of modular functions such as theta functions* and *symmetry groups of the hyperbolic geometry of the upper half-plane* --- all of which are topics (cf. the discussion above of Galois groups versus abstract groups!) that bear a profound relationship to IUT.